Linear Algebra I

30/01/2018, Tuesday, 14:00 – 17:00

You are **NOT** allowed to use any type of calculators.

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1 Linear systems of equations (2 + 3 + 3 + 2 + 2 + 3 = 15 pts)

Consider the linear system of equations

$$\begin{array}{rcrcrcl} 2x & - & y & + & 3z & = & a \\ x & - & y & + & z & = & b \\ 7x & - & 2y & + & 12z & = & c \end{array}$$

in the unknowns x, y, and z.

(a) Write down the corresponding augmented matrix.

(b) Put it into the row echelon form.

(c) Determine the values of a, b, and c such that the system is consistent.

(d) Determine the lead and free variables when the system is consistent.

(e) Find the solution set.

Required Knowledge: **Gauss-elimination, row operations, row echelon form, consistency, and set of solutions.**

Solution:

1a: The augmented matrix is given by

$$\left[\begin{array}{rrr|c} 2 & -1 & 3 & a \\ 1 & -1 & 1 & b \\ 7 & -2 & 12 & c \end{array}\right].$$

1b: By applying row operations, we obtain:

$$\left[\begin{array}{rrr|c} 2 & -1 & 3 & a \\ 1 & -1 & 1 & b \\ 7 & -2 & 12 & c \end{array}\right] \xrightarrow[\textcircled{2} = \textcircled{1}]{\textcircled{1} = \textcircled{2}} \left[\begin{array}{rrr|c} 1 & -1 & 1 & b \\ 2 & -1 & 3 & a \\ 7 & -2 & 12 & c \end{array}\right] \xrightarrow[\textcircled{3} = \textcircled{3} - 7 \cdot \textcircled{1}]{\textcircled{2} = \textcircled{2} - 2 \cdot \textcircled{1}} \left[\begin{array}{rrr|c} 1 & -1 & 1 & b \\ 0 & 1 & 1 & a - 2b \\ 0 & 5 & 5 & c - 7b \end{array}\right]$$

$$\left[\begin{array}{rrr|c} 1 & -1 & 1 & b \\ 0 & 1 & 1 & a - 2b \\ 0 & 5 & 5 & c - 7b \end{array}\right] \xrightarrow{\textcircled{3} = \textcircled{3} - 5 \cdot \textcircled{2}} \left[\begin{array}{rrr|c} 1 & -1 & 1 & b \\ 0 & 1 & 1 & a - 2b \\ 0 & 0 & 0 & c - 5a + 3b \end{array}\right]$$

1c: The system is consistent of and only if $c = 5a - 3b$.

1d: When $c = 5a - 3b$, the lead variables are x and y whereas z is the free variable.

1e: From the echelon form, we see that

$$y = a - 2b - z$$

and

$$x = b + y - z = a - b - 2z.$$

This leads to the general solution

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} a - b \\ a - 2b \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}.$$

2 Determinants (5 + 6 + 4 = 15 pts)

Consider the $n \times n$ matrix of the form

$$B_n = \begin{bmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{bmatrix}.$$

Let $D_n = \det(B_n)$.

(a) Find D_1, D_2, and D_3.

(b) Expand along the first row and find numbers a and b such that $D_n = aD_{n-1} + bD_{n-2}$ for $n \geqslant 3$.

(c) Prove by induction on n that $D_n = n + 1$ for $n \geqslant 3$.

REQUIRED KNOWLEDGE: **Determinants, row/column operations.**

SOLUTION:

2a: Note that

$$B_1 = 2, \qquad B_2 = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \qquad \text{and} \qquad B_3 = \begin{bmatrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

Then, we have

$$D_1 = 2, \qquad D_2 = 3, \qquad \text{and} \qquad D_3 = 4.$$

2b: By expanding along the first row, we get

$$D_n = \begin{vmatrix} 2 & 1 & 0 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & 0 & \cdots & 1 & 2 \end{vmatrix}$$

$$= 2 \cdot \begin{vmatrix} 2 & 1 & 0 & \cdots & 0 & 0 \\ 1 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{vmatrix} - \begin{vmatrix} 1 & 1 & 0 & \cdots & 0 & 0 \\ 0 & 2 & 1 & \cdots & 0 & 0 \\ 0 & 1 & 2 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 2 & 1 \\ 0 & 0 & 0 & \cdots & 1 & 2 \end{vmatrix}$$

Now, we observe that the first determinant on the right hand side equals D_{n-1}. Expanding the second one along the first column, we see that it equals D_2. This means that

$$D_n = 2D_{n-1} - D_{n-2}.$$

2c: From (2b), we already know that $D_3 = 4$. Assume that there exists n such that

$$D_k = k + 1$$

for all k with $3 \leqslant k \leqslant n$. Note that

$$D_{n+1} = 2D_n - D_{n-1} = 2(n+1) - n = n+2.$$

Hence, $D_n = n+1$ for all $n \geqslant 3$.

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3 Vector spaces (7 + 8 = 15 pts)

(a) Let $X \in \mathbb{R}^{n\times n}$ and $S(X) = \{A \in \mathbb{R}^{n\times n} \mid AX + XA = 0\}$. Show that $S(X)$ is a subspace of $\mathbb{R}^{n\times n}$. For $n = 3$ and $X = \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}$, find a basis for $S(X)$ and find its dimension.

(b) Let a and $b \neq 0$ be real numbers. Consider the functions $f(x) = e^{ax}\sin(bx)$, $g(x) = e^{ax}\cos(bx)$ and the subspace $S = \operatorname{span}(f, g)$ of the vector space of continuous functions. Let $L : S \to S$ be given by $L(h) = h''$ where h'' denotes the second derivative of h. Show that L is a linear transformation. Find its matrix representation relative to the basis (f, g).

REQUIRED KNOWLEDGE: **Vector spaces, subspaces, basis, dimension, linear transformations, matrix representations.**

SOLUTION:

3a: To show that $S(X)$ is a subspace, we begin with the observation that $0_{n\times n} \in S(X)$, that is $S(X) \neq \varnothing$. Let α be a scalar and $A \in S(X)$. Note that

$$(\alpha A)X + X(\alpha A) = \alpha(AX + XA) = 0.$$

Hence, we obtain $\alpha A \in S(X)$. This means that $S(X)$ is closed under scalar multiplication. Now, let A and B belong to $S(X)$ and note that

$$(A + B)X + X(A + B) = (AX + XA) + (BX + XB) = 0.$$

Thus, we see that $A + B \in S(X)$. This means that $S(X)$ is closed under vector addition. Consequently, $S(X)$ is a subspace.

Let $n = 3$ and $X = \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}$. Suppose that

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \in S(X).$$

By definition of $S(X)$, we have

$$\begin{aligned} 0 = AX + XA &= \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix} + \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \\ &= \begin{bmatrix} 0 & a_{11} & 0 \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{bmatrix} + \begin{bmatrix} a_{21} & a_{22} & a_{23} \\ 0&0&0\\0&0&0 \end{bmatrix} = \begin{bmatrix} a_{21} & a_{11} + a_{22} & a_{23} \\ 0 & a_{21} & 0 \\ 0 & a_{31} & 0 \end{bmatrix}. \end{aligned}$$

As such, we obtain $a_{11} + a_{22} = 0$, $a_{21} = a_{23} = a_{31} = 0$. Therefore, A must be of the form

$$\begin{aligned} A &= \begin{bmatrix} a & b & c \\ 0 & -a & 0 \\ 0 & d & e \end{bmatrix} \\ &= a\begin{bmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{bmatrix} + b\begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix} + c\begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix} + d\begin{bmatrix} 0&0&0\\0&0&0\\0&1&0 \end{bmatrix} + e\begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}. \end{aligned}$$

Hence, we see that the matrices $\left\{\begin{bmatrix} 1&0&0\\0&-1&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&1\\0&0&0\\0&0&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&1&0 \end{bmatrix}, \begin{bmatrix} 0&0&0\\0&0&0\\0&0&1 \end{bmatrix}\right\}$ span the subspace $S(X)$. Also note that they are linearly independent. This means that they form a basis and the dimension of $S(X)$ is 5.

3b: Let α and β be scalars and h and j belong to S. Note that

$$L(\alpha h + \beta j) = (\alpha h + \beta j)'' = \alpha h'' + \beta j'' = \alpha L(h) + \beta L(j).$$

Therefore, L is a linear transformation.

In order to find the matrix representation, we need to apply the transformation on the basis vectors. First note that

$$\begin{aligned}\big(e^{ax}\sin(bx)\big)' &= a\big(e^{ax}\sin(bx)\big) + b\big(e^{ax}\cos(bx)\big)\\ \big(e^{ax}\cos(bx)\big)' &= -b\big(e^{ax}\sin(bx)\big) + a\big(e^{ax}\cos(bx)\big).\end{aligned}$$

As such, we have

$$\begin{aligned}\big(e^{ax}\sin(bx)\big)'' &= a^2\big(e^{ax}\sin(bx)\big) + ab\big(e^{ax}\cos(bx)\big) + ab\big(e^{ax}\cos(bx)\big) - b^2\big(e^{ax}\sin(bx)\big)\\ &= (a^2-b^2)\big(e^{ax}\sin(bx)\big) + (2ab)\big(e^{ax}\cos(bx)\big)\\ \big(e^{ax}\cos(bx)\big)'' &= -ab\big(e^{ax}\sin(bx)\big) - b^2\big(e^{ax}\cos(bx)\big) + a^2\big(e^{ax}\cos(bx)\big) - ab\big(e^{ax}\sin(bx)\big)\\ &= (-2ab)\big(e^{ax}\sin(bx)\big) + (a^2-b^2)\big(e^{ax}\cos(bx)\big).\end{aligned}$$

Consequently, the matrix representation is given by

$$\begin{bmatrix} a^2-b^2 & -2ab \\ 2ab & a^2-b^2 \end{bmatrix}.$$

4 Least squares problem (15 pts)

Find the parabola of the form $y = a + bx^2$ that gives the best least squares approximation to the points:

x	0	1	2
y	0	2	3

REQUIRED KNOWLEDGE: **Least-squares problem, normal equations.**

SOLUTION:

The corresponding least squares problem is given by

$$\begin{bmatrix} 1 & 0 \\ 1 & 1 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \\ 3 \end{bmatrix}.$$

This leads to the following normal equations:

$$\begin{bmatrix} 3 & 5 \\ 5 & 17 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 5 \\ 14 \end{bmatrix}.$$

Therefore, we obtain the least squares solution as

$$\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 3 & 5 \\ 5 & 17 \end{bmatrix}^{-1} \begin{bmatrix} 5 \\ 14 \end{bmatrix} = \frac{1}{26} \begin{bmatrix} 17 & -5 \\ -5 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ 14 \end{bmatrix} = \begin{bmatrix} \frac{15}{26} \\ \frac{17}{26} \end{bmatrix}.$$

5 Characteristic polynomial, determinant, and trace $(5 + (1 + 2 + 3 + 4) = 15 \text{ pts})$

(a) Let $M \in \mathbb{R}^{n\times n}$ and $p(s) = p_k s^k + p_{k-1} s^{k-1} + \cdots + p_1 s + p_0$ be a polynomial. Show that if λ is an eigenvalue of M then $p(\lambda)$ is an eigenvalue of $p(M) = p_k M^k + p_{k-1} M^{k-1} + \cdots + p_1 M + p_0 I_n$.

(b) Let $A \in \mathbb{R}^{n\times n}$ be such that $A^2 - A + I_n = 0$.

- (i) Show that A is nonsingular.
- (ii) Show that n is an even number.
- (iii) Show that the characteristic polynomial of A is $(\lambda^2 - \lambda + 1)^{\frac{n}{2}}$.
- (iv) Show that $\operatorname{tr}(A) = \frac{n}{2}$ and $\det(A) = 1$.

Required Knowledge: **Characteristic polynomial, eigenvalues, determinant, trace, and determinant.**

Solution:

5a: Suppose that λ is an eigenvalue of M. Then, there must exists a nonzero x such that $Mx = \lambda x$. Note that $M^k x = \lambda^k x$. This means that $p(M)x = p(\lambda)x$. Since x is not zero, we can conclude that $p(\lambda)$ is an eigenvalue of $p(M)$.

5a: alternative Let x be such that $Ax = 0$. Then, we have $0 = (A^2 - A + I)x = x$. Hence, A is nonsingular.

5a: yet another alternative From $A^2 - A + I = 0$, we see that $A - A^2 = I$. This leads to $A(I - A) = I$. Therefore, A is nonsingular and $I - A$ is its inverse.

5b: (i) From (5a), we know that if λ is an eigenvalue of A then $\lambda^2 - \lambda + 1$ must be an eigenvalue of $A^2 - A + I$. Since $A^2 - A + I = 0$, we see that if λ is an eigenvalue of A then $\lambda^2 - \lambda + 1 = 0$. Hence, we can conclude that all eigenvalues of A must be of the form $\frac{1}{2} \pm \frac{\sqrt{3}}{2}$. Therefore, A is nonsingular as zero is not an eigenvalue.

(ii) Suppose that n is odd. Since A is a real matrix, it must have a real eigenvalue. However, we already know from (i) that all eigenvalues of A are nonreal. Therefore, n must be even.

(iii) Since nonreal eigenvalues of a real matrix must occur in conjugate pairs. It follows from (i) that the characteristic polynomial of A is $(\lambda^2 - \lambda + 1)^{\frac{n}{2}}$.

(iv) We know that the trace of a matrix is equal to sum of its eigenvalues whereas its determinant equals the product of its eigenvalues. Note that $(\frac{1}{2} + \frac{\sqrt{3}}{2}) + (\frac{1}{2} - \frac{\sqrt{3}}{2}) = 1$ and $(\frac{1}{2} + \frac{\sqrt{3}}{2})(\frac{1}{2} - \frac{\sqrt{3}}{2}) = 1$. Therefore, it follows from (iii) that $\operatorname{tr}(A) = \frac{n}{2}$ and $\det(A) = 1$.

6 Eigenvalues/vectors (1 + 2 + 4 + 4 + 4 = 15 pts)

Consider the matrix

$$M = \begin{bmatrix} a & 2b & 0 \\ b & a & b \\ 0 & 2b & a \end{bmatrix}$$

where a and $b \neq 0$ are real numbers.

(a) Show that a is an eigenvalue of M.

(b) Find an eigenvector corresponding to the eigenvalue a.

(c) Find other eigenvalues of M and corresponding eigenvectors.

(d) Determine the values of a and b such that M is nonsingular.

(e) Is M diagonalizable? If so, find a diagonalizer.

REQUIRED KNOWLEDGE: **Eigenvalues, eigenvectors, and diagonalization, matrix exponential.**

SOLUTION:

6a: Note that

$$\det(aI - M) = \det(\begin{bmatrix} 0 & 2b & 0 \\ b & 0 & b \\ 0 & 2b & 0 \end{bmatrix}) = 0$$

since the first and the third rows of M are the same. Therefore, a must be an eigenvalue.

6b: We need to solve the homogeneous system

$$0 = (aI - M)x = \begin{bmatrix} 0 & 2b & 0 \\ b & 0 & b \\ 0 & 2b & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Since $b \neq 0$, we see that $x_2 = 0$. Since $bx_1 + bx_3 = 0$, we see that

$$\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

is an eigenvector corresponding to the eigenvalue a.

6c: The characteristic polynomial of M is given by

$$\begin{aligned}\det(\lambda I - M) = \det \begin{bmatrix} \lambda - a & -2b & 0 \\ -b & \lambda - a & -b \\ 0 & -2b & \lambda - a \end{bmatrix} &= (\lambda - a)^3 - 2b^2(\lambda - a) - 2b^2(\lambda - a) \\ &= (\lambda - a)^3 - 4b^2(\lambda - a) \\ &= (\lambda - a)\left((\lambda - a)^2 - 4b^2\right)\end{aligned}$$

Therefore, we see that the eigenvalues are a and $a \pm 2b$.

For the eigenvalue $a + 2b$, the eigenvectors are nontrivial solution of the homogeneous system

$$0 = ((a + 2b)I - M)\, x = \begin{bmatrix} 2b & -2b & 0 \\ -b & 2b & -b \\ 0 & -2b & 2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Then, an eigenvector corresponding to the eigenvalue $a + 2b$ is given by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

For the eigenvalue $a - 2b$, the eigenvectors are nontrivial solution of the homogeneous system

$$0 = ((a - 2b)I - M)\,x = \begin{bmatrix} -2b & -2b & 0 \\ -b & -2b & -b \\ 0 & -2b & -2b \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = 0.$$

Then, an eigenvector corresponding to the eigenvalue $a - 2b$ is given by

$$\begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

6d: We know that the determinant is the product of eigenvalues. Therefore, M is nonsingular if and only if $a \neq 0$ and $a \neq 2b$ and $a \neq -2b$.

6e: Since $b \neq 0$, eigenvalues are distinct. As such, M is diagonalizable. Alternatively, M is diagonalizable since there are three linearly independent eigenvectors. A diagonalizer can be obtained from the eigenvector as follows:

$$\begin{bmatrix} a & 2b & 0 \\ b & a & b \\ 0 & 2b & a \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & a + 2b & 0 \\ 0 & 0 & a - 2b \end{bmatrix}.$$